DIVISION OF THE HUMANITIES AND SOCIAL SCIENCES

CALIFORNIA INSTITUTE OF TECHNOLOGY

PASADENA, CALIFORNIA 91125

FIRST BEST BAYESIAN PRIVATIZATION MECHANISMS

Maciej K. Dudek

Taesung Kim

John O. Ledyard

SOCIAL SCIENCE WORKING PAPER 896

August 1994
Revised April 1995

First Best Bayesian Privatization Mechanisms

Maciej K. Dudek Taesung Kim John O. Ledyard

Abstract

A planner is interested in designing an ex-post efficient, individually rational, Bayesian mechanism for allocating a single indivisible object to one of the agents who knows his own valuation and only the distribution of other agents' valuations of the object. In this paper, we show that it is impossible to design such a mechanism without any transfers among agents and the planner. However, we discover and describe an ex-post efficient, ex-post individually rational, Bayesian mechanism which balances transfers among agents without any payment to (or from) the planner.

Our result that an ex-post efficient, ex-post individually rational, transfer balanced, Bayesian mechanism exists, is in stark contrast to two well–known impossibility results in the literature; the nonexistence of a Bayesian public good mechanism satisfying ex-post efficiency, individual rationality and budget balance (Laffont and Maskin (1979)) and the impossibility of an ex-post efficient, individually rational, Bayesian bilateral trading mechanism between a seller and a buyer without an outside subsidy (Myerson and Satterthwaite (1983)).

First Best Bayesian Privatization Mechanisms*

Maciej K. Dudek[†] Taesung Kim[‡] John O. Ledyard[§]

1 Introduction

Consider a planner who is interested in allocating a single indivisible object (a "prize") to one of several agents in the economy. Each agent knows his own valuation for the object, but only knows the distribution of other agents' valuations. The planner's objective is to find an ex-post efficient mechanism. In other words, a mechanism which always assigns the object to the agent who values it most, while balancing the transfers among agents, so there is no payment (or subsidy) to (or from) the planner. The mechanism also has to be incentive compatible and ex-post individually rational. Each agent is guaranteed a nonnegative gain by participating in the mechanism. We call solutions to this problem *first best privatization mechanisms.*

Two main types of questions regarding privatization mechanisms are examined in this paper. The first type is motivated by the fact that in many applications it is illegal or impossible to compensate agents who participate in the mechanism (see, e.g., Guler, Plott, and Vuong (1994)). We look at the extreme case where no transfers among agents are allowed. We ask whether the planner can design an ex-post efficient, Bayesian-incentive compatible mechanism for allocating the object without any transfers among agents. The answer is no (Theorem 1 in Section 3). Then we ask what mechanism gives the best performance among Bayesian-incentive compatible mechanisms without transfers. We answer this question by showing that no Bayesian-incentive compatible mechanism without transfers can interim Pareto dominate a lottery mechanism. Moreover, we show that

*We are grateful to Kim Border, Leo Hurwicz, Mathew Jackson, Herve Moulin and Tom Palfrey for their comments on an early version. We have also benefitted greatly from the many valuable suggestions of two anonymous referees.

†California Institute of Technology, Pasadena, CA 91125

‡Seoul National University, Seoul, Korea

§California Institute of Technology, Pasadena, CA 91125. He thanks the office of projects design and strategic planning of the Jet Propulsion Labaratory of NASA for their support.

for any Bayesian-incentive compatible mechanism without transfers, there exists a lottery mechanism which weakly interim Pareto dominates this mechanism. (See Theorem 2 in Section 3)

The second type question is concerned with mechanisms where transfers among the agents are allowed, but are required to be balanced (i.e., outside payments to or from the planner are not allowed). Specifically, we ask: Can the planner design an individually-rational, ex-post efficient, Bayesian-incentive compatible mechanism with balanced transfers? The answer is yes. In answering this question we examine mechanisms satisfying the strongest form of individual rationality, namely *ex-post individual rationality*. Specifically, we present a Bayesian mechanism which implements *ex-post* efficient, *ex-post* individually-rational allocations with balanced transfers (Theorem 3 in Section 4). Moreover, if every agent's valuation is drawn from the same distribution, our mechanism strictly interim Pareto dominates the simple equal chance lottery (Theorem 4 in Section 4).

There are many works in the literature using the Bayesian approach to study mechanism design. In the context of public goods economies, D'Aspremont and Gerard-Varet (1979) discovered a Bayesian mechanism that achieves ex-post efficiency and budget balance. However, the mechanism of D'Aspremont and Gerard-Varet is not interim individually rational. Laffont and Maskin (1979) showed that, in general, ex-post efficiency and interim-individual rationality are incompatible in budget balanced, Bayesian public goods mechanisms. Mailath and Postlewaite (1990), Ledyard and Palfrey (1994) and Rob (1989) respectively have shown that in large economies interim-individual rationality and incentive compatibility imply that the public good will never be produced.

In the context of bilateral trading of a single, private indivisible object between one seller and one buyer, Myerson and Satterthwaite (1983) proved the impossibility of ex-post efficient, interim-individually rational, Bayesian mechanisms without an outside subsidy. More recently, however, Makowski and Mezzetti (1993) showed that if in addition to a seller there are at least two potential buyers whose valuations are independently drawn from the *same* distribution, then for *some* distributions of valuations there exist ex-post efficient, interim-individually rational, Bayesian mechanisms for trading the object.

Considering the impossibility theorems predominant in the literature, our possibility results are rather surprising. Our possibility results rely on two main factors. First, our problem is to allocate a single private indivisible object rather than a public good as in Laffont and Maskin (1979).

Second, and more importantly, in the bilateral trading literature (Myerson and Satterthwaite (1983), Makowski and Mezzetti (1993)) the object is owned by one of the agents in the economy—namely the seller—but in our model it is owned by the planner. Our possibility result suggests that the property right, which makes the individual-rationality condition hard to satisfy in a bilateral trading model, is the main obstacle to achieving ex-post efficiency with a Bayesian mechanism. Our result is somewhat consistent with another positive result of Cramton, Gibbons and Klemperer (1987), which shows that a partnership can be dissolved in an ex-post efficient, interim individually rational way if every agent's valuation is independently drawn from the *same* distribution and no partner has too large a share. However, in our model each agent's valuation does not have to be drawn from the same distribution and the problems considered are quite different. Makowski and Mezzetti (1994) also has a result related to ours. For the agents with uniformly-distributed valuations, they proved the existence of first-best, interim IR mechanisms. On the other hand, we present a specific first-best mechanism which works for the general class of distributions. Most importantly, our mechanism is ex-post IR while Cramton, Gibbons and Klemperer (1987) and Makowski and Mezzetti (1994) only required interim IR.

Although the standard in Bayesian mechanism design is *interim*-individual rationality, there are various reasons why we are interested in *ex-post* individually-rational mechanisms. First, with ex-post IR, the mechanism can be operated without relying on external credit markets or outside subsidies which are needed in interim IR mechanisms to avoid the bankruptcy problem. Second, even if external credit markets or outside subsidies are available, in reality the designer often cannot prevent an agent from dropping out of the mechanism ex-post when the final outcome gives him negative utility. Third, most of the rules used by real world institutions, such as auctions, double auctions, bid-ask markets, etc., are ex-post individually rational.

In the mechanism-design literature, Gresik (1991) and Sappington (1983) in a different context use ex-post IR as one of the requirements of the mechanism. Gresik showed how to construct ex-post IR, ex-ante efficient, bilateral trading mechanisms from interim IR, ex-ante efficient mechanisms. Sappington showed that in a principal-agent model between risk-neutral parties, with an ex-post IR constraint, the first-best outcome cannot be achieved, although it can be with an interim IR constraint. In both cases, to achieve ex-post IR, one must sacrifice the ex-post efficiency. However, in our privatization mechanism we are able to achieve both ex-post IR and ex-post efficiency.

In the complete information implementation context, where the agents' valuations of the object are common knowledge among agents, Glazer and Ma (1989) introduced multi-stage mechanisms assigning the object to the agent with the highest valuation without any transfer of money among agents at equilibrium. However, we show that with incomplete information any ex-post efficient, Bayesian mechanism must involve nonzero monetary transfers among agents at equilibrium (Theorem 1 in Section 3). Therefore, our results prove that Glazer and Ma's result does not hold in the case of incomplete information.

The rest of this paper is organized as follows. In Section 2, a formal model is given. In Section 3 we discuss Bayesian incentive compatible mechanisms without transfers.

Then, in Section 4 we present an ex-post efficient, ex-post individually rational, transfer balanced, Bayesian incentive compatible mechanism.

2 The Model

Consider a problem where a planner allocates a single indivisible object (a "prize") to one of $n(\geq 2)$ agents in the economy. Agent i's valuation, v_i, of this object is known only to agent i, but it is common knowledge that v_i is an independent random variable with the distribution function F_i on $[a, b]$, where $[a, b]$ is the set of possible valuations of agent i and $0 \leq a \leq b$. Let f_i be the probability density function corresponding to F_i (so $f_i = F_i'$). We assume that each f_i is positive on its domain $[a, b]$. Every agent in the economy knows the value of the object to the planner, c (≥ 0). We assume that $c = 0$ since the results in this paper can be easily generalized to any positive c. (See Corollary 5.)

These agents participate in the mechanism to determine who receives the object and how much money should be transferred between agents. In a *direct mechanism* all agents report their valuations simultaneously to the planner who then determines the recipient of the object and the amount of monetary transfers between the agents. Such a mechanism is described by outcome functions (p, x) on $[a, b]^n$, where

$$p(v) = (p_1(v), p_2(v), ..., p_n(v)) \quad \text{with} \quad \sum_{i=1}^{n} p_i(v) = 1,\ p_i(v) \geq 0,\ v = (v_1, ..., v_n)$$

are the probabilities that the object will be given to the agent i and

$$x(v) = (x_1(v), x_2(v), ..., x_n(v))$$

are the monetary transfers to agent i when agent i reports v_i.

Let $v_{-i} = (v_1, ..., v_{i-1}, v_{i+1}, ..., v_n)$, and let $E_{-i}(\cdot)$ be the expectation operator with respect to v_{-i}. The direct mechanism (p, x) is called Bayesian *incentive compatible* if each type of each player wants to report truthfully when others report truthfully; for all i,

$$U_i(v_i; v_i) \geq U_i(v_i; \hat{v}_i) \quad \forall v_i, \hat{v}_i \in [a, b],$$

where

$$U_i(v_i; \hat{v}_i) = v_i E_{-i}(p_i(\hat{v}_i, v_{-i})) + E_{-i}(x_i(\hat{v}_i, v_{-i})).$$

$U_i(v_i; \hat{v}_i)$ is the interim expected utility of agent i of type v_i if i reports $\hat{v}_i$. By the revelation principle (e.g., Gibbard (1973), Myerson (1979)), we do not lose any generality by restricting our attention to Bayesian incentive compatible direct mechanisms, which we simply call Bayesian mechanisms from now on.

To guarantee that every agent participates in the mechanism, each agent has to be better off participating than not. In this paper we ask for the mechanism to be *ex-post individually rational*, i.e., $\forall i$, $\forall v \in [a, b]$,

$$v_i p_i(v) + x_i(v) \geq 0.$$

That is, for any realization of valuations v each agent has to receive nonnegative utility in the mechanism. Ex-post individual rationality obviously implies interim individual rationality. In an interim individually rational mechanism, some agents can be worse off ex-post for some realization of valuations v. Some agents may want to walk away from the mechanism after the decision of the mechanism is revealed. However, if the mechanism is ex-post individually rational, no agent has an ex-post incentive to walk away from the mechanism.

Ex-post efficiency requires that, for any realization of types, the object be given to the agent with the highest valuation. Thus, the mechanism is *ex post efficient* if

$$p_i(v) = \begin{cases} 1 & \text{if} \quad v_i > v_k \text{ for all } k \neq i \\ 0 & \text{if} \quad v_i < v_k \text{ for some } k. \end{cases}$$

In case more than one agent has the highest valuation, then the object can be given to any of them in the ex-post efficient mechanism. However, since this happens with probability zero, it is ignored in the above definition of ex-post efficiency.

We introduce two different types of restrictions on the monetary transfers among agents and the planner. The mechanism is *without transfers* if $\forall v$, $\forall i$,

$$x_i(v) = 0.$$

The mechanism is called *transfer balanced* if $\forall v$,

$$\sum_{i=1}^{n} x_i(v) = 0.$$

In a transfer-balanced mechanism, transfers are allowed between agents, but the planner cannot collect (or subsidize) any money from (or to) the set of agents.

3 The Impossibility of a First-Best Mechanism Without Transfers

In this section two questions are addressed regarding mechanisms without transfers. The first question is: Can ex-post efficiency be achieved by some mechanism without any monetary transfers? This type of question was posed and answered positively in the complete information framework by Glazer and Ma (1989). However, if there is incomplete information, the answer is no.

1 Theorem *Let $G_i(t) = \Pi_{k \neq i} F_k(t)$. This is the probability that all v_k other then v_i will be less than or equal to t. If there is some i such that $G_i(\hat{v}_i) \neq G_i(v_i^*)$ for some $\hat{v}_i, v_i \in (a, b)$, then there is no ex-post efficient Bayesian mechanism without transfers.*

Proof: Suppose (p, x) is an ex-post efficient Bayesian mechanism without transfers. Then $x_j(v) = 0$ for all j and all v. Also $p_i(v) = 1$ if $v_i > v_k$ for all $k \neq i$ and $p_i(v) = 0$ if $v_i < v_k$ for some k. Further, in order for (p, v) to be Bayesian incentive compatible, it must be true that for all i $U_i(v_i^*; v_i^*) \geq U_i(v_i^*, \hat{v}_i)$ for all $v_i^*, \hat{v}_i \in [a, b]$. Thus:

$$v_i^* \int p_i(v/v_i^*) dF_{-i}(v_{-i}) \geq v_i^* \int p_i(v/\hat{v}_i) dF_{-i}(v_{-i})$$

and

$$\hat{v}_i \int p_i(v/\hat{v}_i) dF_{-i}(v_{-i}) \geq \hat{v}_i \int p_i(v/v_i^*) dF_i(v_{-i})_R$$

But $\int p_i(v/\hat{v}_i) dF_{-i}(v_{-i}) = G_i(\hat{v}_i)$. Therefore,

$$\hat{v}_i[G_i(\hat{v}_i) - G_i(v_i^*)] \geq 0 \text{ and } v_i^*[G_i(v_i^*) - G_i(\hat{v}_i)] \geq 0.$$

So if there is an ex-post efficient Bayesian mechanism, then for all i and all $\hat{v}_i, v_i^* > 0, G_i(\hat{v}_i) = G_i(v_i^*)$. But by hypothesis, there is an i with $\hat{v}_i, v_i^* > 0$ such that $G_i(\hat{v}_i) \neq G_i(v_i^*)$. ∎

The incomplete information assumption needed to get impossibility is truly minimal: only one i need be unsure whether one of two values yields a higher probability for winning.

3.1 Remark *In Theorem 1 above we showed that no ex-post efficient mechanism exists without transfers. Since Glazer and Ma's multi-stage mechanism (1989) with complete information is without transfers along the equilibrium path, but allows transfers of the equilibrium path, one wonders whether a mechanism without transfers only along the equilibrium path can be designed even in the incomplete information framework. However, Theorem 1 suggests that the answer is no. To see this, suppose there is a mechanism, possibly an extensive form, whose outcome is ex-post efficient and which is transfer balanced along the equilibrium path. Then by the revelation principle there is a direct mechanism in which truth is a Bayesian Nash equilibrium and transfers along the equilibrium are zero. The proof of Theorem 1 shows that it is impossible.*

The second question is: What mechanism gives the best performance without transfers? The answer is that any lottery is one of the best Bayesian mechanisms without transfers that planner can design. We prove this only for two agents since generalization to any number of agents is trivial. The mechanism is called a *lottery* mechanism if $\forall v_1, v_2$,

$$p_1(v_1, v_2) = \alpha, \; p_2(v_1, v_2) = 1 - \alpha, \text{ and } x_1(v_1, v_2) = x_2(v_1, v_2) = 0.$$

for some $0 \leq \alpha \leq 1$. Moreover, it is called the *equal chance lottery* mechanism if it is a lottery mechanism with $\alpha = 1/2$. A lottery mechanism is obviously Bayesian incentive compatible and ex-post individually rational. Even though it is not ex-post efficient, it is one of the best in the following sense:

2 Theorem **i.** *No Bayesian mechanism (p, x) without transfers can interim Pareto dominate a lottery mechanism.*

ii. *For any Bayesian incentive compatible mechanism (p, x) without transfers, there exists a lottery mechanism which weakly interim Pareto dominates (p, x).*

Proof: **i.** For the mechanism (p, x) to satisfy Bayesian incentive compatibility for agent 1, $\forall v_1, \hat{v}_1$,

$$U_1(v_1; v_1) = v_1 \int_a^b p_1(v_1, t_2) f_2(t_2) \; , dt_2 \geq v_1 \int_a^b p_1(\hat{v}_1, t_2) f_2(t_2), dt_2.$$

So, $\forall v_1, \hat{v}_1$,

$$\int_a^b p_1(v_1, t_2) f_2(t_2)\, dt_2 = \int_a^b p_1(\hat{v}_1, t_2) f_2(t_2), dt_2.$$

Similarly, for agent 2, it follows that $\forall v_2, \hat{v}_2$,

$$\int_a^b p_2(t_1, v_2) f_1(t_1), dt_1 = \int_a^b p_2(t_1, \hat{v}_2) f_1(t_1), dt_1.$$

Therefore,

$$\begin{aligned} U_1(v_1; v_1) &= v_1 \int_a^b p_1(v_1, t_2) f_2(t_2)\, dt_2 \\ &= v_1 \int_a^b \left(\int_a^b p_1(v_1, t_2) f_2(t_2)\, dt_2 \right) f_1(v_1)\, dv_1, \end{aligned}$$

and

$$\begin{aligned} U_2(v_2; v_2) &= v_2 \int_a^b p_2(t_1, v_2) f_1(t_1)\, dt_1 \\ &= v_2 \int_a^b \left(\int_a^b p_2(t_1, v_2) f_1(t_1)\, dt_1 \right) f_2(v_2)\, dv_2. \end{aligned}$$

If the mechanism (p, x) interim Pareto dominates the a lottery mechanism for some $\alpha \in [0, 1]$ at (v_1, v_2), then without loss of generality,

$$U_1(v_1; v_1) > \alpha v_1, \quad \text{and} \quad U_2(v_2; v_2) \geq (1 - \alpha) v_2.$$

Then,

$$\begin{aligned} \int_a^b \int_a^b p_1(t_1, t_2) f_2(t_2) f_1(t_1)\, dt_1\, dt_2 &> \alpha, \text{ and} \\ \int_a^b \int_a^b p_2(t_1, t_2) f_1(t_1) f_2(t_2)\ , dt_1\ , dt_2 &\geq 1 - \alpha, \end{aligned}$$

which is a contradiction since $p_1(t_1, t_2) + p_2(t_1, t_2) = 1$, $\forall t_1, t_2$.

ii. Let

$$\alpha_1(v_1) = \int_a^b p_1(v_1, v_2) f_2(v_2)\, dv_2,$$

and

$$\alpha_2(v_2) = \int_a^b p_2(v_1, v_2) f_1(v_1)\, dv_1.$$

Since (p, x) is incentive compatible and without transfers, by the same argument in the proof of **i.** $\alpha_i(v_i)$ is a constant number for all v_i. So, let

$$\alpha_1 = \alpha_1(v_1) \text{ and } \alpha_2 = \alpha_2(v_2).$$

Moreover, $\alpha_1 + \alpha_2 = 1$ since

$$\begin{aligned} \alpha_1 + \alpha_2 &= \int_a^b \alpha_1(v_1) f_1(v_1)\, dv_1 + \int_a^b \alpha_2(v_2) f_2(v_2) \ , dv_1 \\ &= \int_a^b \int_a^b (p_1(v_1, v_2) + p_2(v_1, v_2)) f_1(v_1) f_2(v_2) \ , dv_1 \ , dv_2. \end{aligned}$$

The interim utility $U_1(v_1; v_1)$ for agent 1 under (p, x) is

$$\begin{aligned} U_1(v_1; v_1) &= v_1 \int_a^b p_1(v_1, v_2) f_2(v_2)\, dv_2 \\ &= \alpha_1 v_1. \end{aligned}$$

Therefore, for every agent the interim utility from (p, x) is the same as that from the lottery mechanism with the winning probability α_1 to the agent 1. ∎

3.2 Remark *A Bayesian mechanism with the property* **i** *in Theorem 2 is called interim incentive efficient in Holmstrom and Myerson (1983).*

If transfers between agents are allowed but are required to be balanced, then as we show in the next section, not only ex-post efficiency but also ex-post individual rationality can be achieved via a Bayesian mechanism. Moreover, if the distribution of each agent's valuation is same (i.e., $F_i = F$ for all i), then the mechanism we propose interim Pareto dominates the equal chance lottery mechanism.

4 Ex-Post Individually Rational, First Best Mechanism

In this section, we introduce a simple transfer balanced, Bayesian mechanism which implements ex-post efficient, ex-post individually rational allocations. Since we require transfer balance on the part of the planner, auction mechanisms, such as the second price auction in which the seller extracts money from the buyers, cannot be considered as candidates. However, we suggest the following simple mechanism.

For the analysis of this section, we define the new distribution functions G, G_i, and G_{ij} from F_i's as follows; for all t,

$$\begin{aligned} G(t) &= \prod_{k=1}^{n} F_k(t) \\ G_i(t) &= \prod_{k \neq i} F_k(t); \text{and} \\ G_{ij}(t) &= \prod_{k \neq i,j} F_k(t). \end{aligned}$$

3 Theorem *The mechanism defined by*

$$p_i(v) = \begin{cases} 1 & \textit{if } v_i \geq v_k \ \forall k \quad \textit{and} \ \ i = \min\{1 \leq j \leq n : \ v_j \geq v_k \ \forall k\} \\ 0 & \textit{otherwise} \end{cases}$$

$$\begin{aligned} x_i(v) &= -\sum_{k \neq i} \int_a^{v_i} \frac{\int_a^t G(s)\, ds}{G(t)} \frac{f_k(t)}{F_k(t)}\, dt - \frac{n-1}{n} a \\ & \qquad \textit{if } v_i \geq v_k \ \forall k \ \textit{and}\ i = \min\{1 \leq j \leq n : \ v_j \geq v_k \ \forall k\} \\ &= \int_a^{v_j} \frac{\int_a^t G(s), ds}{G(t)} \frac{f_i(t)}{F_i(t)}\, , dt + \frac{1}{n} a \ \textit{if } j (\neq i) \textit{ is a winner.} \end{aligned}$$

is Bayesian incentive compatible, transfer balanced, ex-post efficient and ex-post individually rational.

4.1 Remark *In the case that two or more agents report the highest valuation, the above mechanism assigns the object to the agent with the lowest index among them. However, since ties occur with probability zero, they can be ignored in the analysis below.*

Proof: If agent i reports $\hat{v}_i$, then the probability that he gets the object is $G_i(\hat{v}_i)$. In this event, agent i gets the transfer of

$$-\sum_{k\neq i}\int_a^{\hat{v}_i} \frac{\int_a^t G(s)\,ds}{G(t)} \cdot \frac{f_k(t)}{F_k(t)}\,dt - \frac{n-1}{n}a.$$

If agent k reports the highest valuation v_k, then agent i ($i \neq k$) receives the transfer of

$$\int_a^{v_k} \frac{\int_a^t G(s)\,ds}{G(t)} \cdot \frac{f_i(t)}{F_i(t)}\,dt + \frac{1}{n}a.$$

Note that $G_{ik}(v_k)$ is the probability that every agent except agents i, k have valuation lower than v_k. Therefore,

$$\begin{aligned} U_i(v_i;\hat{v}_i) &= G_i(\hat{v}_i)\cdot\left(v_i - \sum_{k\neq i}\int_a^{\hat{v}_i} \frac{\int_a^t G(s)\ ,ds}{G(t)} \cdot \frac{f_k(t)}{F_k(t)}\ ,dt - \frac{n-1}{n}a\right) \\ &+\sum_{k\neq i}\int_{v_k=\hat{v}_i}^{b} G_{ik}(v_k)f_k(v_k)\cdot\left(\int_a^{v_k} \frac{\int_a^t G(s)\ ,ds}{G(t)} \cdot \frac{f_i(t)}{F_i(t)}\ ,dt + \frac{1}{n}a\right)\ ,dv_k. \end{aligned} \tag{1}$$

To show that this mechanism is Bayesian incentive compatible, note that for all i,

$$\begin{aligned} \frac{\partial U_i(v_i;\hat{v}_i)}{\partial \hat{v}_i} &= G_i'(\hat{v}_i)\cdot\left(v_i - \sum_{k\neq i}\int_a^{\hat{v}_i} \frac{\int_a^t G(s)\,ds}{G(t)} \cdot \frac{f_k(t)}{F_k(t)}\,dt - \frac{n-1}{n}a\right) \\ &-G_i(\hat{v}_i)\cdot\sum_{k\neq i}\frac{\int_a^{\hat{v}_i} G(s)\,ds}{G(\hat{v}_i)}\cdot\frac{f_k(\hat{v}_i)}{F_k(\hat{v}_i)} - \sum_{k\neq i} G_{ik}(\hat{v}_i)f_k(\hat{v}_i)\cdot\left(\int_a^{\hat{v}_i} \frac{\int_a^t G(s)\,ds}{G(t)} \cdot \frac{f_i(t)}{F_i(t)}\,dt + \frac{1}{n}a\right) \\ &= \sum_{k\neq i} G_{ik}(\hat{v}_i)f_k(\hat{v}_i)\cdot\left(v_i - \sum_{k\neq i}\int_a^{\hat{v}_i} \frac{\int_a^t G(s)\,ds}{G(t)} \cdot \frac{f_k(t)}{F_k(t)}\,dt - \frac{n-1}{n}a\right) \end{aligned}$$

$$-\sum_{k\neq i} G_{ik}(\hat{v}_i) f_k(\hat{v}_i) \frac{\int_a^{\hat{v}_i} G(s)\,ds}{G(\hat{v}_i)} - \sum_{k\neq i} G_{ik}(\hat{v}_i) f_k(\hat{v}_i) \cdot \left(\int_a^{\hat{v}_i} \frac{\int_a^t G(s)\,ds}{G(t)} \cdot \frac{f_i(t)}{F_i(t)}\,dt + \frac{1}{n} a \right)$$

$$= \sum_{k\neq i} G_{ik}(\hat{v}_i) f_k(\hat{v}_i) \cdot \left(v_i - \sum_{k\neq i} \int_a^{\hat{v}_i} \frac{\int_a^t G(s)\,ds}{G(t)} \cdot \frac{f_k(t)}{F_k(t)}\,dt - \frac{\int_a^{\hat{v}_i} G(s)\,ds}{G(\hat{v}_i)} \right.$$

$$\left. - \int_a^{\hat{v}_i} \frac{\int_a^t G(s)\,ds}{G(t)} \cdot \frac{f_i(t)}{F_i(t)}\,dt - a \right)$$

$$= \sum_{k\neq i} G_{ik}(\hat{v}_i) f_k(\hat{v}_i) \cdot \left(v_i - a - \frac{\int_a^{\hat{v}_i} G(s)\,ds}{G(\hat{v}_i)} - \sum_{k=1}^{n} \int_a^{\hat{v}_i} \frac{\int_a^t G(s)\,ds}{G(t)} \cdot \frac{f_k(t)}{F_k(t)}\,dt \right)$$

$$= \sum_{k\neq i} G_{ik}(\hat{v}_i) f_k(\hat{v}_i) \cdot \left(v_i - a - \frac{\int_a^{\hat{v}_i} G(s)\,ds}{G(\hat{v}_i)} + \left. \frac{\int_a^t G(s)\,ds}{G(t)} \right|_a^{\hat{v}_i} - \int_a^{\hat{v}_i} dt \right)$$

by using integration by parts

$$= \sum_{k\neq i} G_{ik}(\hat{v}_i) f_k(\hat{v}_i) \cdot (v_i - \hat{v}_i) \text{ since } \lim_{t\to a} \frac{\int_a^t G(s)\,ds}{G(t)} = \lim_{t\to a} \frac{G(t)}{G'(t)} = 0.$$

Notice also that

$$\begin{aligned} \frac{\partial U_i(v_i;\hat{v}_i)}{\partial \hat{v}_i} &> 0 \quad \text{if} \quad \hat{v}_i < v_i, \\ \frac{\partial U_i(v_i;\hat{v}_i)}{\partial \hat{v}_i} &= 0 \quad \text{if} \quad \hat{v}_i = v_i, \\ \frac{\partial U_i(v_i;\hat{v}_i)}{\partial \hat{v}_i} &< 0 \quad \text{if} \quad \hat{v}_i > v_i. \end{aligned}$$

So, $U_i(v_i;\hat{v}_i)$ is maximized if $\hat{v}_i = v_i$, hence the mechanism is Bayesian incentive compatible.

By the definition of (p,x), it is easy to see that the mechanism is ex-post efficient and transfer balanced. Moreover, the mechanism is ex-post individually rational since the transfer to the loser is always nonnegative and

$$\sum_{k\neq i} \int_a^{v_i} \frac{\int_a^t G(s)\,ds}{G(t)} \cdot \frac{f_k(t)}{F_k(t)}\,dt + \frac{n-1}{n} a$$

$$
\begin{aligned}
&\leq \sum_{k\neq i}\int_a^{v_i} \frac{\int_a^t G_i(s)\,ds}{G_i(t)}\cdot\frac{f_k(t)}{F_k(t)}\,dt + \frac{n-1}{n}a \qquad \text{since } \frac{F_i(s)}{F_i(t)} \leq 1\ \forall s \leq t \\
&= \int_a^{v_i} \frac{\int_a^t G_i(s)\,ds}{G_i(t)} \sum_{k\neq i}\frac{f_k(t)}{F_k(t)}\,dt + \frac{n-1}{n}a \\
&= \int_a^{v_i} \frac{\int_a^t G_i(s)\,ds}{G_i(t)}\cdot\frac{G_i'(t)}{G_i(t)}\,dt + \frac{n-1}{n}a \\
&= -\frac{\int_a^t G_i(s)\,ds}{G_i(t)}\Bigg|_a^{v_i} + \int_a^{v_i} dt + \frac{n-1}{n}a \text{ by integration by parts} \\
&= v_i - a - \frac{\int_a^{v_i} G_i(s)\,ds}{G_i(v_i)} + \frac{n-1}{n}a \text{ since } \lim_{t\to a}\frac{\int_a^t G_i(s)\,ds}{G_i(t)} = \lim_{t\to a}\frac{G_i(t)}{G_i'(t)} = 0 \\
&= v_i - \frac{1}{n}a - \frac{\int_a^{v_i} G_i(s)\,ds}{G_i(v_i)} \\
&\leq v_i,
\end{aligned}
$$

meaning that winner's net transfer is less than the his valuation of the object. ∎

Example Consider the case where two agents' distributions are given by

$$
\begin{aligned}
F_1(v_1) &= v_1^\alpha, \\
F_2(v_2) &= v_2^\beta
\end{aligned}
$$

on $[0,1]$ for some α, $\beta > 0$. Then for this environment our mechanism in Theorem 3 turns out to be

$$
p_1(v) = \begin{cases} 1 & \text{if } v_1 \geq v_2 \\ 0 & \text{otherwise} \end{cases}
$$

$$
x_1(v) = \begin{cases} -\frac{\beta}{\alpha+\beta+1}v_1 & \text{if } v_1 \geq v_2 \\ \frac{\alpha}{\alpha+\beta+1}v_2 & \text{otherwise} \end{cases}
$$

A particularly interesting and simple example arises for the case of identical uniform distributions ($\alpha = \beta = 1$). Here the agent with the highest value gets the item and pays 1/3 of his value to the other agent. If there are n agents, the highest valued agent with value v pays $\frac{1}{n+1}v$ to each of the other $n-1$ agents. Thus, the winner gets $\frac{2}{n+1}v$ and the losers each get $\frac{1}{n+1}v$. Notice that everyone is better off than 0 (ex-post individually rational), transfers balance since

$$\frac{2}{n+1}v + \frac{n-1}{n+1}v = v,$$

and the item goes to the highest value (first best allocation). Finally, it is easy to show that it is Bayesian incentive compatible.

4.2 Remark *Note that our mechanism may admit another Bayesian Nash equilibrium with less desirable outcomes. However, Palfrey and Srivastava (1991) showed that for independent types and private values any incentive efficient allocation can be uniquely implemented by an augmentation of a direct mechanism.*

4.3 Remark *For the finite type case, the requirements of ex-post individual rationality and Bayesian incentive compatibility with ex-post efficiency can be expressed simply as a system of linear inequalities. Kim and Ledyard (1994) provided mechanisms for the finite type case based on a Theorem of Alternative.*

In Section 3 we showed that among the class of mechanisms without transfers any lottery mechanism is interim incentive efficient. Therefore, if each agent's valuation is drawn from the same distribution, i.e., $F_i = F$ for all i, the equal chance mechanism seems to be a reasonable candidate without transfers. However, if we allow transfers among agents, we have the following result:

4 Theorem *If $F_i = F$ for all i and F is strictly increasing, then the mechanism defined in Theorem 3 interim Pareto dominates the equal chance mechanism, i.e., $\forall i$, $\forall v_i$,*

$$U_i(v_i; v_i) > \frac{1}{n}v_i.$$

Proof: Since $F_i = F$ for all i, by (1)

$$
\begin{aligned}
U_i(v_i; v_i) &= \left(v_i - \sum_{k \neq i} \int_a^{v_i} \frac{\int_a^t (F(s))^n \, ds}{(F(t))^{n+1}} f(t) \, dt - \frac{n-1}{n} a \right) \cdot (F(v_i))^{n-1} \\
&\quad + \sum_{k \neq i} \int_{v_k = v_i}^{b} (F(v_k))^{n-2} f(v_k) \cdot \left(\int_a^{v_k} \frac{\int_a^t (F(s))^n \,, ds}{(F(t))^{n+1}} f(t), dt + \frac{1}{n} a \right) dv_k.
\end{aligned}
$$

Let

$$
\begin{aligned}
W_i(v_i) &= U_i(v_i; v_i) - \frac{1}{n} v_i, \\
H(v_i) &= (F(v_i))^{n-1}\,; \quad \text{and} \\
h(v_i) &= H'(v_i).
\end{aligned}
$$

Then,

$$
h(v_i) = (n-1)(F(v_i))^{n-2} f(v_i) = \frac{(n-1) f(v_i) H(v_i)}{F(v_i)}.
$$

Then,

$$
\begin{aligned}
W(v_i) &= (F(v_i))^{n-1} \cdot \left(v_i - (n-1) \int_a^{v_i} \frac{\int_a^t (F(s))^n \, ds}{(F(t))^{n+1}} f(t) \, dt - \frac{n-1}{n} a \right) \\
&\quad + (n-1) \int_{v = v_i}^{b} (F(v))^{n-2} f(v) \cdot \left(\int_a^{v} \frac{\int_a^t (F(s))^n \,, ds}{(F(t))^{n+1}} f(t) \,, dt + \frac{1}{n} a \right) \,, dv - \frac{1}{n} v_i \\
&= H(v_i) \cdot \left(v_i - (n-1) \int_a^{v_i} \frac{\int_a^t (F(s))^n \, ds}{(F(t))^{n+1}} f(t) \, dt - \frac{n-1}{n} a \right) \\
&\quad + \int_{v = v_i}^{b} h(v) \cdot \left(\int_a^{v} \frac{\int_a^t (F(s))^n \, ds}{(F(t))^{n+1}} f(t) \, dt + \frac{1}{n} a \right) dv - \frac{1}{n} v_i.
\end{aligned}
$$

Notice that

$$
\begin{aligned}
W'(v_i) &= H(v_i)\cdot(1-(n-1)\frac{\int_a^{v_i}(F(s))^n\,,ds}{(F(v_i))^{n+1}}f(v_i)) \\
&\quad +h(v_i)\cdot\left(v_i-(n-1)\int_a^{v_i}\frac{\int_a^t(F(s))^n\,ds}{(F(t))^{n+1}}f(t)\,dt-\frac{n-1}{n}a\right) \\
&\quad -h(v_i)\cdot\left(\int_a^{v_i}\frac{\int_a^t(F(s))^n\,ds}{(F(t))^{n+1}}f(t)\,,dt+\frac{1}{n}a\right)-\frac{1}{n} \\
&= H(v_i)-\frac{1}{n}+h(v_i)\cdot(v_i-\frac{\int_a^{v_i}(F(s))^n\,,ds}{(F(v_i))^n}-n\int_a^{v_i}\frac{\int_a^t(F(s))^n\,,ds}{(F(t))^{n+1}}f(t)\,,dt-a) \\
&= H(v_i)-\frac{1}{n}\text{ by using the integration by parts.}
\end{aligned}
$$

Choose v_i^* satisfying $H(v_i^*)=\frac{1}{n}$. Since W_i is a convex function,

$$W_i(v_i)\geq W_i(v_i^*)$$

for all v_i. Moreover,

$$
\begin{aligned}
W_i(v_i^*) &> H(v_i^*)\cdot\left(v_i^*-(n-1)\int_a^{v_i^*}\frac{\int_a^t(F(s))^n\,,ds}{(F(t))^{n+1}}f(t)\,,dt-\frac{n-1}{n}a\right) \\
&\quad +\int_{v=v_i^*}^b h(v)\cdot\left(\int_a^{v_i^*}\frac{\int_a^t(F(s))^n\,,ds}{(F(t))^{n+1}}f(t)\,,dt+\frac{1}{n}a\right)\,,dv-\frac{1}{n}v_i^* \\
&= H(v_i^*)\cdot\left(v_i^*-(n-1)\int_a^{v_i^*}\frac{\int_a^t(F(s))^n\,,ds}{(F(t))^{n+1}}f(t)\,,dt-\frac{n-1}{n}a\right) \\
&\quad +\left(\int_a^{v_i^*}\frac{\int_a^t(F(s))^n\,ds}{(F(t))^{n+1}}f(t)\,dt+\frac{1}{n}a\right)\cdot(1-H(v_i^*))-\frac{1}{n}v_i^* \\
&= \frac{1}{n}v_i^*-\int_a^{v_i^*}\frac{\int_a^t(F(s))^n\,,ds}{(F(t))^{n+1}}f(t)\,dt\cdot\left(-\frac{n-1}{n}+1-\frac{1}{n}\right) \\
&\quad -\frac{1}{n}a\cdot\left(-\frac{n-1}{n}+1-\frac{1}{n}\right)-\frac{1}{n}v_i^* \\
&= 0.
\end{aligned}
$$

Therefore, for all v_i

$$U_i(v_i; v_i) > \frac{1}{n} v_i.$$

∎

The following Corollary shows that the assumption that the value of the object to the planner, c, is 0 is innocuous if the planner is only interested in collecting c from the agents whenever the object is given to one of the agents. Notice that for $c(\geq 0)$, ex-post efficiency means that if any agent's valuation is higher than c, the one with the highest valuation receives the object and that if every agent's valuation is lower than c, then the planner keeps it. Also, in this context transfer balance requires that the planner collect exactly c from the agents. We first consider the following modified problem; let

$$w_i = v_i - c, \;\; \tilde{F}_i(t) = pr[w_i \leq t | w_i \geq 0],$$

where v_i is the valuation of agent i with the distribution function F_i. Then $\tilde{F}_i$ is defined on $[0, b-c]$. Let (p, x) be the same mechanism given in Theorem 3 except that agent i's distribution is $\tilde{F}_i$ instead of F_i for each i.

Define the new mechanism $(\tilde{p}, \tilde{x})$ as follows;

i.

$$\tilde{p}_i(v) = \begin{cases} 1 & \text{if } v_i \geq v_k, \ \forall k \text{ and } i = \min\{1 \leq j \leq n : \ v_j \geq v_k \ \forall k\} \text{ and } v_i \geq c \\ 0 & \text{otherwise} \end{cases}$$

ii. the planner keeps the object if $v_k < c$ for all k,

iii. the planner collects c from the winner if there is any winner, and

iv.

$$\tilde{x}_i(v) = \begin{cases} x_i(w) & \text{if } v_i \geq c \\ 0 & \text{if } v_i < c \end{cases}$$

where $w = (w_1, \ldots, w_n)$ and $w_j = \max\{v_j - c, 0\} \forall j$.

5 Corollary *For $c \geq 0$, the mechanism above is Bayesian incentive compatible, ex-post efficient, ex-post individually rational, and transfer balanced.*

We finish with a couple of comments comparing our results to others.

4.4 Remark *Compared to the Myerson and Satterthwaite's (1983) finding of the impossibility of an ex-post efficient, individually rational, trading mechanism between a seller who initially owns the object and a buyer, our possibility result relies on the fact that the property right to the object is not given to one of the agents but to the outside planner. Therefore, it is easier for individual rationality to be satisfied in our framework. This implies that, contrary to the Coase theorem, achieving an efficient outcome heavily depends on the assignment of the property right in the incomplete information framework.*

One might conjecture that a lottery allocation of the property right followed by an after-market or sale by the new owner of the right, might be equivalent to our mechanism. But again from Myerson and Satterthwaite (1983) and Gresik and Satterthwaite (1989) we know that unless the property right happens to go to the highest value agent, first best allocations will not be achieved. Thus our mechanism interim dominates not only the equal chance lottery mechanism (Theorem 4) but also that mechanism followed by an after-market since the property right creates a monopoly which the government, in our model, does not take advantage of.

4.5 Remark *In public goods economies, it is well known that ex-post efficient, budget balanced, Bayesian mechanisms cannot always be interim individually rational (see Laffont and Maskin (1979)). If we interpret the planner's value c in our model as the cost of producing the object, the situation of allocating a single indivisible private good is drastically different from that of allocating a public good. Corollary 5 shows that one can design ex-post efficient, ex-post individually rational, budget balanced Bayesian mechanisms for allocating a private good.*

One of the themes in Groves and Ledyard (1987) was that, in classical economies with a finite number of agents, there is no distinction between private and public goods in designing incentive compatible, ex-post efficient, individually rational mechanisms if the incentive compatibility concept is either complete information Nash equilibrium or dominant strategy equilibrium. Our results show that with a finite number of agents, there is a distinction between private and public goods in designing Bayesian incentive compatible, ex-post efficient, individually rational mechanisms.

4.6 Remark *There remain at least two interesting open questions. We have not provided a characterization of all interim incentive efficient mechanisms subject to ex-post individual rationality and balanced transfers. (We do provide a characterization for no transfers in Theorem 1.)*

We also have not considered the case where agents' valuations are correlated. In the case of public goods mechanisms with incomplete information, D'Aspremont, Cremer and

Gerard-Varet (1993) identified the conditions under which ex-post efficiency and budget balance can be achieved under correlation. Those techniques may also apply to private goods.

References

[1] D'Aspremont, C., Cremer, J. and Gerard-Varet, L.-A. 1993. Correlation, independence, and Bayesian implementation. mimeographed.

[2] Cramton, P., Gibbons, R. and Klemperer, P. 1987. Dissolving a partnership efficiently. *Econometrica* 55: 615-632.

[3] D'Aspremont, C., and Gerard-Varet, L.-A. 1979. Incentives and incomplete information. *Journal of Public Economics* 11: 25-45.

[4] Gibbard, A. 1973. Manipulation of voting schemes. *Econometrica* 41: 587-602.

[5] Glazer, J. and Ma, C. A. 1989. Efficient allocation of a "prize" – King Solomon's dilemma. *Games and Economic Behavior* 1: 222-233.

[6] Gresik, T. A. 1991. Ex-ante efficient, ex-post individually rational trade. *Journal of Economic Theory* 53: 131-145.

[7] Gresik, T. A. and Satterthwaite, M. A. 1989. The rate at which a simple market converges to efficiency as the number of traders increases: an asymptotic result for optimal trading mechanisms. *Journal of Economic Theory* 48: 304-332.

[8] Groves, T. and Ledyard, J. O. 1987. Incentive compatibility since 1972, in *Information, Incentives and Economic Mechanisms.* T. Groves, R. Radner and S. Reiter Eds. University of Minnesota Press.

[9] Guler, K., Plott, C. and Vuong, Q. 1994. A study of zero-out auctions: testbed experiments of a process of allocating private rights to the use of public property. *Economic Theory* 4: 67-104.

[10] Holmstrom, B. and Myerson, R. 1983. Efficient and durable decision rules with incomplete information. *Econometrica* 51: 1799-1819.

[11] Kim, T. and Ledyard, J. O. 1994. First Best Bayesian Privatization Mechanisms. California Institute of Technology Working Paper #896.

[12] Laffont, J.-J. and Maskin, E. 1979. A differential approach to expected utility maximizing mechanisms, in *Aggregation and Revelation of Preferences.* J.-J. Laffont Ed. North-Holland, Amsterdam.

[13] Ledyard, J. and Palfrey, T. 1994. Voting and lottery draft as efficient public goods mechanisms. *Review of Economic Studies* 61: 327-355.

[14] Mailath, G. and Postlewaite, A. 1990. Asymmetric information bargaining problems with many agents. *Review of Economic Studies* 57: 351-367.

[15] Makowski, L. and Mezzetti, C. 1993. The possibility of efficient mechanisms for trading an indivisible object. *Journal of Economic Theory* 59: 451-465.

[16] Makowski, L. and Mezzetti, C. 1994. Bayesian and Weakly Robust First Best Mechanisms: Characterizations. *Journal of Economic Theory* 64: 500-519.

[17] Myerson, R. 1979. Incentive compatibility and the bargaining problem. *Econometrica* 47: 61-73.

[18] Myerson, R., and Satterthwaite, M. 1983. Efficient mechanisms for bilateral trading. *Journal of Economic Theory* 28: 265-281.

[19] Palfrey, T. and Srivastava, S. 1991. Efficient trading mechanisms with pre-play communication. *Journal of Economic Theory* 55: 17-40.

[20] Rob, R. 1989. Pollution claim settlements with private information. *Journal of Economic Theory* 47: 307-333.

[21] Sappington, D. 1983. Limited liability contracts between principal and agent. *Journal of Economic Theory* 29: 1-21.